International Journal of Theoretical Physics, Vol. 7, No. 5 (1973), pp. 379–388

Symmetric Lie Algebras of Non-Linear Transformations of Conformal Type in Quantum Mechanics

HANS TILGNER

Institut für theoretische Physik (I), der Universität,
355 Marburg, West Germany, Renthof 7

Received: 13 *June* 1972

Abstract

The Koecher construction of simple symmetric Lie algebras is used to realize colineation and conformal Lie algebras of non-linear transformations of a pseudo-orthogonal vector space in the canonical Weyl algebras, which are used in the Schrödinger representation. The realization maps the linear sub-algebras onto symmetrized polynomials of second degree, whereas the non-linear parts are mapped onto polynomials of first and third degree. For the two examples the Meyberg Jordan algebras are explicitly given.

1. *Introduction*

Lie algebras are of great practical use for the construction of spectra of dynamical systems in quantum mechanics, if they are realized as invariance Lie algebras in the associative algebra of observables. But even if not all elements commute with the Hamiltonian, in certain cases the Lie algebra can be used as a non-invariance Lie algebra for the determination of the spectrum. Thus the embedding of a given Lie algebra into the algebra of observables is of general interest.

In the present article we study embeddings into the Weyl algebra, which is the polynomial algebra generated by the position and momentum operators of quantum mechanics. The explicit calculations are done in the Schrödinger representation, which is known to be the only irreducible one up to equivalence of the Weyl algebra with a finite number of p and q's.

The Lie algebras to be studied are the colineation and conformal Lie algebras as examples of a class of simple Lie algebras, namely those which admit a symmetric decomposition (Section 2). In physics they are realized by non-linear transformations of a (in general pseudo-) orthogonal vector space, cf. equation (6.2), which is the Euclidean three-dimensional space in the case of colineations or the Minkowski space. These non-linear realizations originate in the use of the corresponding groups in geometry and kinematics, for instance the special colineations

$$C_a(x) = \frac{x}{1 + \langle a, x \rangle}$$

in geometrical optics and the special conformal transformations

$$K_a(x) = \frac{x + \langle x, x \rangle a}{1 + 2\langle x, a \rangle + \langle a, a \rangle \langle x, x \rangle}$$

in electrodynamics. Therefore, the underlying spaces of the non-linear realizations have a physical meaning whereas the vector spaces of the self representations—equations (2.12) and (2.13), i.e. the smallest faithful representation spaces—have none.

It is shown that the symmetric decomposition corresponds to a decomposition of the Lie algebras of transformations according to their degree of homogeneity (zero, one or two). The decomposition gives rise to a –2 graduation which can be found in certain subspaces of the Weyl algebra as well, if one decomposes it into spaces of symmetrized polynomials of equal degree. Although the whole Weyl algebra carries no such Lie graduation, it can be found in the spaces of first, second and third degree, if one considers only abelian subspaces of first and third degree. For the explicit construction of a graduation preserving realization the method of pairings by M. Koecher is used.

In a certain sense the above construction is a generalization of the embedding into polynomials of second degree. Since they contain a Lie algebra which is isomorphic to the general linear algebra, the theorem of Ado allows such a 'linear' realization for all Lie algebras. However, to realize simple symmetric Lie algebras in that way one has to introduce unphysical p and q's, since the self-representation spaces have more dimensions than the non-linear realization spaces. Thus the minimalizing corresponds to the dropping of the linearity requirement, but depends on a symmetric decomposition.

Notation. We write $ab - ba = [a, b]_- = ad(a)b$ and $2[a, b]_+ = ab + ba$ in associative algebras. Bilinear forms are assumed to be non-degenerate.

2. *Symmetric Decomposition of Simple Lie Algebras*

Given a finite dimensional simple Lie algebra $\mathscr{L}$ over $\mathbb{R}$, the direct vector space decomposition

$$\mathscr{L} = \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3 \tag{2.1}$$

is called symmetric (Koecher, 1969a, p. 59, 1969b, p. 389) if

(S.1) $[\mathscr{L}_1, \mathscr{L}_3] = \mathscr{L}_2$

(S.2) $[\mathscr{L}_2, \mathscr{L}_1] \subset \mathscr{L}_1$

(S.3) $[\mathscr{L}_1, \mathscr{L}_1] = \{0\}$

(S.4) there is an involutive automorphism ϑ of $\mathscr{L}$ with $\vartheta\mathscr{L}_1 = \mathscr{L}_3$

An immediate consequence of the definition is

$$[\mathscr{L}_2, \mathscr{L}_2] \subset \mathscr{L}_2 \tag{2.2}$$

(using the Jakobi identity) and

$$[\mathscr{L}_2, \mathscr{L}_3] \subset \mathscr{L}_3, \qquad [\mathscr{L}_3, \mathscr{L}_3] = \{0\} \tag{2.3}$$

$$\vartheta\mathscr{L}_3 = \mathscr{L}_1, \qquad \vartheta\mathscr{L}_2 \subset \mathscr{L}_2 \tag{2.4}$$

Obviously, $\mathscr{L} = \mathscr{L}_2 \oplus (\mathscr{L}_1 \oplus \mathscr{L}_3)$ is a Cartan decomposition of $\mathscr{L}$ (Loos, 1969, p. 145). The decomposition (2.1) is a special case of

$$\mathscr{L} = \mathscr{L}_{\nu-1} \oplus \mathscr{L}_\nu \oplus \mathscr{L}_{\nu+1}, \qquad \nu = 0, 1, 2, \ldots \tag{2.5}$$

$$[\mathscr{L}_i, \mathscr{L}_k] \subset \mathscr{L}_{i+k-\nu}, \qquad i, k = \nu - 1, \nu, \nu + 1 \tag{2.6}$$

for $\nu = 2$. Here $\mathscr{L}_{\nu-2} = \mathscr{L}_{\nu+2} = \{0\}$. Koecher uses $\nu = 1$. Kobayashi & Nagano (1964) have given a complete classification of symmetric decompositions of real simple Lie algebras, where they use $\nu = 0$. In the following for the embedding into the Weyl algebra, $\nu = 2$ is used because the latter allows such a –2 Lie graduation on certain subspaces.

For $a \oplus M \oplus \vartheta b \in \mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3$ we define transformations sd( ): $\mathscr{L}_1 \to \mathscr{L}_1$ by

$$\mathrm{sd}(a)(c) = a \tag{2.7}$$

$$\mathrm{sd}(M)(c) = [M, c] \tag{2.8}$$

$$\mathrm{sd}(\vartheta a)(c) = \tfrac{1}{2}[[\vartheta a, c], c] \tag{2.9}$$

i.e. in general notation

$$\mathrm{sd}(\mathscr{L}_i)(\mathscr{L}_1) = \frac{(-1)^i}{(i-1)!}\mathrm{ad}(\mathscr{L}_1)^{i-1}\mathscr{L}_i \tag{2.10}$$

The transformations $\mathrm{sd}(\mathscr{L}_i)$ are homogeneous of degree $i - 1$. Continuing these transformations linearly, the definition

$$\{\mathrm{sd}(A), \mathrm{sd}(B)\} = \mathrm{sd}([A, B]), \qquad A, B \in \mathscr{L} \tag{2.11}$$

makes the vector space of these transformations a Lie algebra which is isomorphic to $\mathscr{L}$.

Let us give two matrix examples. For a pseudo-orthogonal vector space $(V, \langle,\rangle)$ we denote the matrix of $\langle,\rangle$ in some basis of V by I, by $\vec{\alpha}$ the column of $a \in V$ and by $\vec{\alpha}^t$ the corresponding row. For $a, b \in V$ and $G \in \mathrm{gl}(V, \mathbb{R})$ the decomposition

$$\begin{pmatrix} 0 & 0 \\ \vec{\alpha} & 0 \end{pmatrix} \oplus \begin{pmatrix} -\dfrac{\operatorname{trace} G}{1+n} & 0 \\ 0 & G - \dfrac{\operatorname{trace} G}{1+n} id_V \end{pmatrix} \oplus \begin{pmatrix} 0 & \vec{\beta}^t I \\ 0 & 0 \end{pmatrix} \tag{2.12}$$

of $\mathrm{sl}(\mathbb{R} \oplus V, \mathbb{R})$ is symmetric with the involutive automorphism $\vartheta G = -G^t$, $G \in \mathrm{sl}(\mathbb{R} \oplus V, \mathbb{R})$.

The second example is given by the pseudo-orthogonal Lie algebra $\mathrm{der}(\tilde{V}, \nless, \ngtr)$ on $\tilde{V} := \mathbb{R} \oplus V \oplus \mathbb{R}$, i.e. the set of $n+2$-dimensional square matrices with $\nless \tilde{A}\tilde{x}, \tilde{y} \ngtr + \nless \tilde{x}, \tilde{A}\tilde{y} \ngtr = 0$, $\tilde{x}, \tilde{y} \in \tilde{V}$, where the (symmetric, invertible) matrix $\tilde{I}$ of $\nless, \ngtr$ is given by $\mathrm{diag}(1, I, -1)$. In matrix form the defining condition for $\tilde{A}$ is $\tilde{A}^t \tilde{I} + \tilde{I}\tilde{A} = 0$. Then the decomposition of $\mathrm{der}(\tilde{V}, \nless, \ngtr)$

$$\begin{pmatrix} 0 & -\vec{\alpha}^t I & 0 \\ \vec{\alpha} & 0 & \vec{\alpha} \\ 0 & \vec{\alpha}^t I & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 & \gamma \\ 0 & M & 0 \\ \gamma & 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \vec{\beta}^t I & 0 \\ -\vec{\beta} & 0 & \vec{\beta} \\ 0 & \vec{\beta}^t I & 0 \end{pmatrix} \tag{2.13}$$

with $\gamma \in \mathbb{R}$ and $M \in \mathrm{der}(V, \langle, \rangle)$, is symmetric with the involutive automorphism $\vartheta\tilde{A} = -\mathbf{I}^{-1}\tilde{A}^t\mathbf{I}$, where $\mathbf{I} := \mathrm{diag}(1, I, 1)$.

The corresponding Lie algebras of transformations are the well-known Lie algebras of infinitesimal colineations and infinitesimal conformal transformations respectively.

The mapping $(a, b) \mapsto a \square b := \mathrm{ad}([a, \vartheta b])|_{\mathscr{L}_1}$ is a *pairing* of $\mathscr{L}_1$, i.e. the mapping $\square\colon \mathscr{L}_1 \times \mathscr{L}_1 \to \mathrm{end}(\mathscr{L}_1)$ satisfies

(P.1) the *trace form* $\mathrm{trace}(a \square b + b \square a)$ is non-degenerate

(P.2) $(a \square b)c = (c \square b)a$

(P.3) the adjoint $(a \square b)^+$ of $a \square b$ with respect to the trace form is $b \square a$

(P.4) $[M, a \square b] = Ma \square b - a \square M^+ b \quad$ for $M \in \mathscr{L}_2$

Here (P.4) shows that $\mathscr{L}_1 \square \mathscr{L}_1$ generates the Lie algebra $\mathscr{L}_2$. Conversely, every such pairing on a vector space V gives rise to a symmetric Lie algebra $V \oplus \mathscr{T} \oplus V$, where $\mathscr{T} \subset \mathrm{end}(V)$ (Koecher, 1969a, Section II3).

The trace forms for the two matrix examples are $-2(n+1)\langle a, b\rangle$ and $-2n\langle a, b\rangle$.

3. *The Schrödinger Representation of the Canonical Commutation Relations*

Let $\mathscr{F}(V)$ be some space of real or complex valued functions on V such that for $f \in \mathscr{F}(V)$ the functions $S(a^*)f$ and $S(a)f$, given by

$$[S(a^*)f](c) = \langle a, c\rangle f(c) \tag{3.1}$$

$$a, c \in V$$

$$[S(a)f](c) = \Delta_c^a f(c) \tag{3.2}$$

(Segal, 1968, p. 148), are in $\mathscr{F}(V)$ again. Here $a^* \in V^*$, i.e. $a \mapsto a^*$ is the isomorphism of V onto its dual space V^* given by $a^*(c) = \langle a, c\rangle$, and $\Delta_c{}^a$ is the directional derivative in the direction of a. If $[,]_-$ denotes the commutator, then an easy verification gives the *canonical commutation relations*

$$[S(a \oplus b^*), S(c \oplus d^*)]_- = (\langle a, d\rangle - \langle c, b\rangle)\, id_{\mathscr{F}(V)} \qquad (3.3)$$

In the following we write $\mathbf{a} = S(a)$ and $\mathbf{a}^* = S(a^*)$. Obviously, $V \oplus V^*$, when equipped with the skew symmetric non-degenerate bilinear form on the right-hand side of (3.3)—which we write σ—is a symplectic vector space, and $\mathbf{V} \oplus \mathbf{V}^*$ generates a representation of the canonical Weyl algebra (Dixmier, 1968; Doebner & Melsheimer, 1968; Nouaze & Revoy, 1970; Tilgner, 1970, 1971). A short description of the Weyl algebra is given below.

The definition $\langle \mathbf{a}, \mathbf{b}\rangle = \langle a, b\rangle$ transports the bilinear forms to $\mathbf{V}$, $\mathbf{V}^*$ and $\mathbf{V} \oplus \mathbf{V}^*$.

4. *The Weyl Algebra of a Symplectic Vector Space*

Let us start from an arbitrary symplectic vector space (E, σ) of dimension $2n$ with non-degenerate bilinear form σ. Given the associative tensor algebra ten(E) over E, we denote by $\otimes$ its multiplication, by 1_t its identity element, and by $((\sigma(x, y)1_t - x \otimes y + y \otimes x))$ the two-sided ideal spanned by all elements of the form $X \otimes (\sigma(x, y)1_t - x \otimes y + y \otimes x) \otimes Y$ for $x, y \in E \subset \text{ten}(E)$ and $X, Y \in \text{ten}(E)$. The associative algebra

$$\text{weyl}(E, \sigma) = \text{ten}(E)/((\sigma(x, y)1_t - x \otimes y + y \otimes x)) \qquad (4.1)$$

is called *Weyl algebra* over (E, σ). If 1_w denotes its identity,

$$xy - yx = \sigma(x, y)1_w, \qquad x, y \in E \subset \text{weyl}(E, \sigma) \qquad (4.2)$$

cf. (3.3). The Weyl algebra is an infinite dimensional, central simple algebra (Nouaze & Revoy, 1970).

A basis of the Weyl algebra is given by 1_w and the symmetrized monomials of the basis elements of E (cf. Tilgner, 1970, 1971). Let $\wedge W_i$ be the $\binom{2n+i-1}{i}$-dimensional vector space spanned by all symmetrized monomials of degree i. Then for all choices of the $x_k \in E \subset \text{weyl}(E, \sigma)$

$$\wedge x_1 x_2, \ldots, x_i = \frac{1}{i!}\sum_{\gamma \in \gamma_i} x_{\gamma(1)} x_{\gamma(2)}, \ldots, x_{\gamma(i)} \qquad (4.3)$$

(γ_i is the permutation group of i objects) is in $\wedge W_i$ and

$$\text{weyl}(E, \sigma) = \bigoplus_{i \geqslant 0} \wedge W_i \qquad (4.4)$$

with $\wedge W_0 = \mathbb{R}1_w$, $\wedge W_1 = E$. For $x_k \in E$ one proves by induction

$$\wedge x_1, \ldots, x_i = [\ldots[x_1, x_2]_+ \ldots]_+, x_i]_+ \qquad (4.5)$$

Hence the vector space $\wedge W_i$ is spanned by the i times iterated anticommutators of elements of E. An analogous statement holds for the

Clifford algebra over a pseudo-orthogonal vector space (Jacobson, 1961, p. 231). One proves

$$[\wedge W_{i+1}, E]_- \subset \wedge W_i \tag{4.6}$$

$$[\wedge W_2, \wedge W_i]_- \subset \wedge W_i \tag{4.7}$$

However, these relations cannot be generalized to a –2 Lie graduation of the Lie algebra weyl(E, σ) because of

$$[\wedge W_3, \wedge W_3]_- \subset \wedge W_4 \oplus \mathbb{R}1_w \tag{4.8}$$

For the proof of this statement one calculates

$$[\wedge x_1 x_2 x_3, \wedge y_1 y_2 y_3]_- = \wedge[x_1 x_2 x_3, y_1 y_2 y_3]_- \oplus \omega 1_w \tag{4.9}$$

where ω is a complicated sum of products of the $\sigma(x_i, y_k)$.

Equations (4.2), (4.6), (4.7) and (4.8) show that a realization $\mathscr{L}_1 \oplus \mathscr{L}_2 \oplus \mathscr{L}_3 \to E \oplus \wedge W_2 \oplus \wedge W_3$ preserves the –2 Lie graduation if $\mathscr{L}_1$ resp. $\mathscr{L}_3$ are mapped into abelian subspaces of E resp. $\wedge W_3$.

5. *The Schrödinger Realization of Symmetric Lie Algebras*

For the realization of the colineation and conformal Lie algebras as polynomials in the **a** and **a***, we use the Koecher construction of symmetric Lie algebras given by Koecher (1969a, Section II3, 1969b). The polynomials

$$g^\Delta(a, b) = -[\mathbf{a}, \mathbf{b}^*]_+ \tag{5.1}$$

$$o^\Delta(a, b) = \mathbf{a}^* \mathbf{b} - \mathbf{b}^* \mathbf{a} = -o^\Delta(b, a) \tag{5.2}$$

are in $\wedge W_2$ with $\mathrm{ad}(g^\Delta(a, b))|_\mathbf{V} \in \mathrm{gl}(\mathbf{V}, \mathbb{R})$ and $\mathrm{ad}(o^\Delta(a, b))|_\mathbf{V} \in \mathrm{der}(\mathbf{V}, \langle,\rangle)$. Let d^Δ be an element of weyl(E, σ) with

$$\mathrm{ad}(d^\Delta)\mathbf{a} = -\mathbf{a}, \qquad \mathrm{ad}(d^\Delta)\mathbf{a}^* = \mathbf{a}^* \tag{5.3}$$

i.e. the $2n \times 2n$ matrix

$$\mathrm{ad}(d^\Delta)|_{\mathbf{V}\oplus\mathbf{V}^*} = \begin{pmatrix} -id_\mathbf{V} & 0 \\ 0 & id_{\mathbf{V}^*} \end{pmatrix}$$

is in the symplectic matrix Lie algebra. Since this Lie algebra is isomorphic to the Lie algebra $\wedge W_2$ (Tilgner, 1971) there is a unique element in $\wedge W_2$ with (5.3), i.e. which is mapped by the isomorphism $\mathrm{ad}(\ )|_{\mathbf{V}\oplus\mathbf{V}^*}: \wedge W_2 \to \mathrm{sp}(\mathbf{V} \oplus \mathbf{V}^*, \sigma)$ into this matrix. If we drop the restriction to $\mathbf{V} \oplus \mathbf{V}^*$ we can add a multiple of 1_w to this element without affecting (5.3).

The mappings $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \square \mathbf{b}$, defined by

$$\mathbf{a} \square \mathbf{b} = \mathrm{ad}(-g^\Delta(a, b) + \langle a, b\rangle d^\Delta)|_\mathbf{V} \tag{5.4}$$

$$\mathbf{a} \square \mathbf{b} = \mathrm{ad}(o^\Delta(a, b) + \langle a, b\rangle d^\Delta)|_\mathbf{V} \tag{5.5}$$

are pairings of **V**. Next we have to look for polynomials $Z(\mathbf{c})$ such that for every $\mathbf{c} \in \mathbf{V}$

$$\mathrm{ad}([\mathbf{a}, Z(\mathbf{c})]_-)|_V = \mathbf{a} \square \mathbf{c} \tag{5.6}$$

in both cases. For this note that there is—up to an additive multiple of 1_w—a unique polynomial q^Δ with

$$\mathrm{ad}(q^\Delta)\,\mathbf{a} = -2\mathbf{a}^*, \qquad \mathrm{ad}(q^\Delta)\,\mathbf{a}^* = 0 \tag{5.7}$$

since the $2n \times 2n$ matrix

$$\begin{pmatrix} 0 & -2id_{\mathbf{V}} \\ 0 & 0 \end{pmatrix}$$

is in $\mathrm{sp}(\mathbf{V} \oplus \mathbf{V}^*, \sigma) = \mathrm{der}(\mathbf{V} \oplus \mathbf{V}^*, \sigma)$. For $\mathbf{c} \in \mathbf{V}$ define

$$\begin{aligned} \delta C_c{}^\Delta &= -[d^\Delta, \mathbf{c}^*]_+ \\ \delta K_c{}^\Delta &= [q^\Delta, \mathbf{c}]_+ - 2[d^\Delta, \mathbf{c}^*]_+ \end{aligned} \tag{5.8}$$

using then the well-known identities

$$\begin{aligned} [uv, w]_- &= u[v, w]_- + [u, w]_- v \\ [[u, v]_+, w]_- &= [[u, w]_-, v]_+ + [u, [v, w]_-]_+ \end{aligned} \tag{5.9}$$

which are valid in any associative algebra, one verifies the commutation relations

$$\begin{aligned} [\delta C_a{}^\Delta, \delta C{}^\Delta]_- &= [\mathbf{a}, \mathbf{b}]_- = 0 \\ [\delta C_a{}^\Delta, \mathbf{b}]_- &= -g^\Delta(a, b) + \langle a, b\rangle\, d^\Delta \\ [\delta C_a{}^\Delta, g^\Delta(c, d)]_- &= \delta C^\Delta{}_{\langle a,\, c\rangle\, d} \\ [\mathbf{a}, g^\Delta(b, c)]_- &= -\langle a, c\rangle\, \mathbf{b} \\ [g^\Delta(b, a), g^\Delta(d, c)]_- &= \langle a, d\rangle\, g^\Delta(b, c) - \langle b, c\rangle\, g^\Delta(d, a) \end{aligned} \tag{5.10}$$

In the second case the commutation relations are

$$\begin{aligned} [\delta K_a{}^\Delta, \delta K_b{}^\Delta]_- &= [\mathbf{a}, \mathbf{b}]_- = [d^\Delta, o^\Delta(a, b)]_- = 0 \\ [\delta K_a{}^\Delta, \mathbf{b}]_- &= 2\{o^\Delta(a, b) + \langle a, b\rangle\, d^\Delta\} \\ [\delta K_a, d^\Delta]_- &= -\delta K_a{}^\Delta \\ [\delta K_a{}^\Delta, o^\Delta(b, c)]_- &= \langle a, b\rangle\, \delta K_c{}^\Delta - \langle a, c\rangle\, \delta K_b{}^\Delta \\ [\mathbf{a}, o^\Delta(b, c)]_- &= \langle a, b\rangle\, \mathbf{c} - \langle a, c\rangle\, \mathbf{b} \\ [o^\Delta(b, a), o^\Delta(d, c)]_- &= \langle a, c\rangle\, o^\Delta(d, b) + \langle b, d\rangle\, o^\Delta(c, a) \\ &\quad - \langle a, d\rangle\, o^\Delta(c, b) - \langle b, c\rangle\, o^\Delta(d, a) \end{aligned} \tag{5.11}$$

together with the first equation (5.3). The last equations in (5.10) and (5.11) are the well-known commutation relations of the Lie algebras $\mathrm{gl}(\mathbf{V}, \mathbb{R})$ and $\mathrm{der}(\mathbf{V}, \langle,\rangle)$, whereas the second equations show that the elements (5.8) have the required property (5.6) of $Z(\mathbf{c})$. Equations (5.10) and (5.11) show that the vector spaces

$$\begin{aligned} &\mathbf{V} \oplus \text{linear span of } g^\Delta(V, V) \oplus \delta C_V{}^\Delta \\ &\mathbf{V} \oplus \text{linear span of } o^\Delta(V, V) + \mathbb{R}d^\Delta \oplus \delta K_V{}^\Delta \end{aligned} \tag{5.12}$$

are Lie algebras, which due to their construction by pairings are symmetric in these decompositions. A verification shows for the matrices (2.12) and

(2.13) the same commutation relations. Thus we have for every choice of d^{Δ} and q^{Δ}, according to (5.3) and (5.7), the colineation and conformal Lie algebras, realized in weyl(E, σ) in a symmetric decomposition.

This construction of an embedding of a symmetric Lie algebra $\mathscr{L}$ into the Weyl algebra can be applied to any symmetric Lie algebra. The difficult question to find a realization of the subspace $\mathscr{L}_3$ by polynomials of third order is then reduced to two easy steps: (i) realize the induced pairing of $\mathscr{L}_1$ on $\mathbf{V} \subset E$, and (ii) find $Z(\mathbf{c})$ with (5.6). The resulting commutation relations then clearly are those of $\mathscr{L}$.

6. *Non-linear Transformations of E Defined by the Weyl Algebra*

Given $Y \in \wedge W_{m+1}$ and $x, y \in E \subset$ weyl(E, σ), we define

$$\mathrm{wd}(1_w)(x) = x, \qquad \mathrm{wd}(y)(x) = y$$

$$\mathrm{wd}(Y)(x) = \frac{1}{m}[\ldots[Y, x]_{-}\ldots]_{-} \qquad (m \text{ times}) \tag{6.1}$$

Continuating this linearly, it follows from (4.6) that the map $x \mapsto \mathrm{wd}(Y)(x)$ of E into itself (for any Y) is homogeneous of degree m for $m \geqslant 0$ and of degree 1 if $m = -1$; thus only the transformations $\mathrm{wd}(1_w)$ and $\mathrm{wd}(\wedge W_2) = \mathrm{ad}(\wedge W_2)|_E$ are linear.

To remove the arbitrariness in the definition of d^{Δ} and q^{Δ} let us demand that the transformations $\mathrm{wd}(Y)$, with Y in the colineation resp. conformal Lie algebra, when restricted to $\mathbf{V}$ coincide with the $\mathrm{sd}(Y)$ for the symmetric realization (5.12). Then from (6.1), d^{Δ} and q^{Δ} must be in $\wedge W_2$, i.e. totally symmetrized, because if one would add a multiple of $id_{\mathscr{F}(V)}$, then $\delta C_a{}^{\Delta}$ and $\delta K_a{}^{\Delta}$ would have an additive term in E, resulting in a zero homogeneous part in the corresponding transformation $\mathrm{sd}(Z(\mathbf{c}))$. The transformations become explicitly

$$\begin{aligned} \mathrm{sd}(\mathbf{a})(\mathbf{c}) &= \mathbf{a}, \quad \mathrm{sd}(d^{\Delta})(\mathbf{c}) = -\mathbf{c} \\ \mathrm{sd}(g^{\Delta}(a, b))(\mathbf{c}) &= \langle b, c\rangle\, \mathbf{a} \\ \mathrm{sd}(o^{\Delta}(a, b))(\mathbf{c}) &= \langle b, c\rangle\, \mathbf{a} - \langle a, c\rangle\, \mathbf{b} \\ \mathrm{sd}(\delta C_a{}^{\Delta})(\mathbf{c}) &= -\langle a, c\rangle\, \mathbf{c} \\ \mathrm{sd}(\delta K_a{}^{\Delta})(\mathbf{c}) &= \langle c, c\rangle\, \mathbf{a} - 2\langle a, c\rangle\, \mathbf{c} \end{aligned} \tag{6.2}$$

From (4.5) it is clear that the elements (5.8) can be written in the totally symmetrized form

$$\begin{aligned} \delta C_c{}^{\Delta} &= -\wedge \mathbf{c}^* d^{\Delta} \\ \delta K_c{}^{\Delta} &= \wedge(\mathbf{c} q^{\Delta} - 2\mathbf{c}^* d^{\Delta}) \end{aligned} \tag{6.3}$$

Thus we have a graduation preserving embedding

$$\mathbf{V} \oplus [\mathbf{V}, Z(\mathbf{V})]_{-} \oplus Z(\mathbf{V}) \subset E \oplus \wedge W_2 \oplus \wedge W_3 \tag{6.4}$$

of both Lie algebras. Using the relation

$$\wedge xyz = xyz - \tfrac{1}{2}\{\sigma(x, y) z - \sigma(z, x) y + \sigma(y, z) x\} \tag{6.5}$$

for $x, y, z \in E \subset \mathrm{weyl}(E, \sigma)$, one can drop the symmetrization $\wedge$ in (6.3), introducing instead an additive term of $\mathbf{V}$. If one introduces a basis $p^1, \ldots, p^n$ in V such that the matrix of $\langle , \rangle$ becomes $\mathrm{diag}(1, \ldots, 1, -1, \ldots, -1)$, then

$$\begin{aligned} d^{\Delta} &= \gamma_{kl}\, \mathbf{q}_k\, \mathbf{p}^l + \frac{n}{2} id_{\mathscr{F}(V)}, \qquad q^{\Delta} = \gamma_{kl}\, \mathbf{q}_k\, \mathbf{q}_l \\ \delta C^{\Delta}_{p^l} &= -\mathbf{q}_i\, \gamma_{kl}\, \mathbf{q}_k\, \mathbf{p}^l - \tfrac{1}{2}(n+1)\, \mathbf{q}_l \\ \delta K^{\Delta}_{p^l} &= \gamma_{kl}\, \mathbf{q}_k\, \mathbf{q}_l\, \mathbf{p}^i - 2\mathbf{q}_i\, \gamma_{kl}\, \mathbf{q}_k\, \mathbf{p}^l - n\mathbf{q}_l \end{aligned} \tag{6.6}$$

where $q_1, \ldots, q_n$ is the induced basis of V^* and γ_{ik} the Kronecker symbol of $\langle , \rangle$ (summation convention!). This form of d^{Δ}, $\delta C^{\Delta}_{p^l}$ and $\delta K^{\Delta}_{p^l}$ differs from the usual form of the generators, calculated by the Lie-Cartan technique, by the last terms, which have thus found a quantum mechanical interpretation. $\delta K^{\Delta}_{p^l}$ and d^{Δ} were used in this form already in the theory of relativistic field equations by Flato *et al.* (1970, p. 86).

The transformations wd resp. their restrictions sd to $\mathbf{V}$ give a non-linear realization of the subspaces $\mathscr{L}_1$ and $\mathscr{L}_3$ of a symmetric Lie algebra $\mathscr{L}$. $\mathrm{sd}(\mathscr{L}_1)$ clearly are the infinitesimal translations on $\mathbf{V}$. These non-linear realizations, together with those of the corresponding groups (given for the special colineations and conformal transformations in the introduction), will be discussed elsewhere for the two examples. It remains to find the non-linear group realizations for other symmetric Lie algebras of Kobayashi and Nagano and to discuss their physical implications.

7. *Related Algebraic Structures*

We add some remarks on a class of algebraic structures, associated with symmetric decompositions.

The pair of vector spaces $(\mathbf{V}, Z(\mathbf{V}))$ is a *Meyberg pair* (Meyberg, 1970, p. 61) for the Lie algebras $[\mathbf{V}, Z(\mathbf{V})]_-$ resp. $\mathscr{L}_2$, and the mapping $S\colon \mathbf{V} \times Z(\mathbf{V}) \to \mathrm{ad}([\mathbf{V}, Z(\mathbf{V})]_-)$, $S\colon (\mathbf{a}, Z(\mathbf{c})) \mapsto \mathrm{ad}([\mathbf{a}, Z(\mathbf{c})]_-)$; i.e. (i) $\mathbf{V}$ and $Z(\mathbf{V})$ are faithful representation spaces of the Lie algebras $[\mathbf{V}, Z(\mathbf{V})]_-$ with respect to ad, (ii) S fulfills the condition (2.2) of Meyberg, and (iii) S generates $[\mathbf{V}, Z(\mathbf{V})]_-$.

To every Meyberg pair there is a class of Jordan algebras with the multiplication $\frac{1}{2}[[\mathbf{a}, Z(\mathbf{c})]_-, \mathbf{b}]_-$, which in the two examples is

$$\{\mathbf{a}, \mathbf{b}\}_{\mathbf{c}} = \langle a, c\rangle \mathbf{b} + \langle b, c\rangle \mathbf{a}$$
$$\{\mathbf{a}, \mathbf{b}\}_{\mathbf{c}} = \langle a, c\rangle \mathbf{b} + \langle b, c\rangle \mathbf{a} - \langle a, b\rangle \mathbf{c}$$

respectively. The second Jordan algebra is known in physics in a slightly modified form: If $\langle e, e\rangle \neq 0$, then $\langle e, e\rangle^{-1}\mathbf{e}$ is the identity element. Choose $\mathbf{V} = \mathbf{V}_0 \oplus \mathbb{R}e$ such that the direct sum is orthogonal with respect to $\langle , \rangle$ and $\langle e, e\rangle = 1$. For $\mathbf{b} = \mathbf{b}_0 \oplus \beta\mathbf{e}$, $\mathbf{c} = \mathbf{c}_0 \oplus \gamma e$, the second Jordan composition is

$$\{\mathbf{b}_0 \oplus \beta\mathbf{e}, \mathbf{c}_0 \oplus \gamma\mathbf{e}\}_e = \beta c_0 + \gamma b_0 \oplus (\beta\gamma - \langle b_0, c_0\rangle)\, \mathbf{e}$$

and this is the Jordan algebra which generates the Clifford algebra on $(\mathbf{V}_0, \langle , \rangle)$. If $\mathbf{b}$ is invertible, then $\mathbf{b}^{-1} = (\beta^2 + \langle b_0, b_0 \rangle)^{-1}(-\mathbf{b}_0 \oplus \beta e)$, i.e. $\mathbf{b}$ is invertible iff $\langle b, b \rangle \neq 0$. Thus every element outside the light cone is invertible. The set of invertible elements of a Jordan algebra is a symmetric space with the multiplication of Loos (1969, p. 68), which becomes here

$$a.b = \frac{1}{\langle b, b \rangle} 2\{\langle a, b \rangle \mathbf{a} - \langle a, a \rangle \mathbf{b}\}$$

The connectivity component which contains $\mathbf{e}$ is the interior of the forward light cone, which is a symmetric space itself. The mass shells, defined by $\langle a, a \rangle =$ constant, are symmetric subspaces.

Acknowledgement

This work was supported in part by the Deutsche Forschungsgemeinchaft and is a part of the author's thesis at the University of Marburg. The author is indebted to K. Meyberg for the communication of the work of Kobayashi and Nagano.

References

Dixmier, J. (1968). *Bulletin de la Société mathématique de France*, **96**, 209.
Doebner, H. D. and Melsheimer, O. (1968). *Journal of Mathematical Physics*, **9**, 1638.
Flato, M., Simon, J. and Sternheimer, D. (1970). *Annals of Physics*, **61**, 78.
Jacobson, N. (1961). *Lie Algebras*. Interscience, New York.
Kobayashi, S. and Nagano, T. (1964). *Journal of Mathematics and Mechanics*, **13**, 875.
Koecher, M. (1969a). *An Elementary Approach to Bounded Symmetric Domains*. Rice University Press, Houston, Texas.
Koecher, M. (1969b). *Mathematische Zeitschrift*, **109**, 349.
Loos, O. (1969). *Symmetric Spaces I: General Theory*. Benjamin, New York.
Meyberg, K. (1970). *Mathematische Zeitschrift*, **115**, 58.
Nouaze, Y. and Revoy, P. (1973). *Bulletin de la Société mathematique de France*, **96**, 27.
Segal, I. (1968). *Topology*, **7**, 147.
Tilgner, H. (1970). *Annales de l'Institut Henri Poincaré*, **A13**, 103.
Tilgner, H. (1971). *Annales de l'Institut Henri Poincaré*, **A14**, 179.